JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 25, No. 6, November–December 2002

Strictly Positive Real $\mathcal{H}_2$ Controller Synthesis via Iterative Algorithms for Convex Optimization

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The synthesis of strictly positive real $\mathcal{H}_2$ controllers for collocated control of large space structures is addressed. To perform this synthesis, a convex optimization technique has already been developed using linear matrix inequalities. This existing technique is based on two design criteria: strict positive realness and the use of $\mathcal{H}_2$ norm as the criterion for optimization. It adopts a common Lyapunov solution to both of these criteria, which results in undesirable conservatism. To reduce this conservatism, a new synthesis technique based on iterative algorithms that can produce superior, noncommon Lyapunov solutions is proposed. Even if a common Lyapunov solution is infeasible, the proposed technique can yield feasible, strictly positive real $\mathcal{H}_2$ controllers. An illustrated example is included.

Nomenclature

$H_{zw}(s)$	=	closed-loop transfer function from w to z
J_{LQG}	=	linear quadratic Gaussian (LQG) cost
$\mathcal{K}_{\mathrm{SPR}}$	=	set of all controller gains K such that $C_K(s) \in \mathcal{P}_c$, where $C_K(s)$ is the transfer function of the observer-based controller defined in Eq. (7)
$M > 0$ $(M < 0)$	=	symmetric positive (negative) definite
$M \geq 0$ $(M \leq 0)$	=	symmetric positive (negative) semidefinite
$M^{1/2}$ for $M \geq 0$	=	unique symmetric square root of M
M'	=	transpose
$\mathcal{P}_c$	=	set of all strictly positive real (SPR) transfer functions
$\mathcal{P}^*$	=	neighbor centered at a solution (P_2^*, P_s^*)
$\delta(\,)$	=	Dirac delta function
$\mathcal{E}$	=	expectation operator
$\|\,\|_2$	=	$\mathcal{H}_2$ norm

I. Introduction

CONTROLLERS of large space structures are usually designed based on reduced-order plant models. However, actual plants are of infinite order theoretically and of very high order in practice. In addition to being mathematically complex, those plants inherently include parameter uncertainties. Therefore, when reduced-order controllers are connected to actual plants, undesired oscillations can occur.[1,2]

To avoid these oscillations, strictly positive real controllers are very useful. It is known that a strictly positive real controller connected to a positive real plant by standard negative feedback can stabilize the closed-loop system. For flexible structural systems (with no zero-frequency or rigid-body modes), the transfer function between collocated rate sensors and force actuators is positive real and is independent of other structural parameters. Hence, any strictly positive real controller is guaranteed to stabilize such an (ideal) structural system. This implies that such a controller designed

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Received 10 December 2001; revision received 17 July 2002; accepted for publication 29 July 2002.

using a truncated modal model is guaranteed to stabilize the physical system. However, strict positive realness alone is not sufficient to achieve good performance, for example, minimizing propellant consumption in space missions. This motivates strictly positive real linear quadratic Gaussian (LQG) or $\mathcal{H}_2$ controllers.[1,3–6]

Lozano-Leal and Joshi[3] and Haddad et al.[4] have provided a sufficient condition on the LQG design weights such that the resulting LQG controller will be strictly positive real. This condition is that a common Lyapunov function be adopted for two design criteria. Shimomura and Fujii[5] have considered the same problem from the viewpoint of the inverse problem, in which two distinct Lyapunov functions are allowed. Geromel and Gapski[6] minimize the $\mathcal{H}_2$ norm of the closed-loop system under the constraint of strict positive realness on controllers. Observer-based controllers were assumed, and the observer gain was predetermined based on noise covariance matrices. The state-feedback gain was then determined using linear matrix inequalities (LMIs). To "convexify" the joint problem of strict positive realness and $\mathcal{H}_2$ optimization, the authors of Ref. 6 have adopted a common Lyapunov solution to both LMI constraints, but this results in excessive conservatism. For further details and other related topics, we refer the reader to Refs. 1, 4, and 6, which include good summaries.

In this paper, we focus on the problem considered in Ref. 6 and attempt to solve the same problem with noncommon Lyapunov solutions to reduce the conservatism of the existing result. Our approach is a combination of completing the square in terms of the controller variable and successive overbounding of negative-semidefinite quadratic terms. It explores a new synthesis technique that allows for noncommon Lyapunov solutions. An illustrated example is included that demonstrates that $\mathcal{H}_2$ performance can be significantly improved.

II. Problem Formulation

Let us consider the following linear time-invariant generalized plant:

$$\Sigma_P : \begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B_1 w + B_2 u \\ z = C_1 \boldsymbol{x} + D_{12} u \\ y = C_2 \boldsymbol{x} + D_{21} w \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) \in \boldsymbol{R}^n$ is the state vector, $u(t) \in \boldsymbol{R}^m$ is the control input, $y(t) \in \boldsymbol{R}^m$ is the measurement output, $w(t) \in \boldsymbol{R}^{m_w}$ is the disturbance input with zero mean and covariance $\mathcal{E}[w(t)w'(\tau)] = \delta(t-\tau)I$, and $z(t) \in \boldsymbol{R}^{m_z}$ is the controlled output. All matrices in Eq. (1) are of compatible dimensions. The following standard assumptions are made:

1) (A, B_1) is controllable and (C_1, A) is observable.
2) (A, B_2) is controllable and (C_2, A) is observable.
3) $D_{12}'C_1 = 0$ and $D_{12}'D_{12} =: R_2 > 0$.
4) $D_{21}B_1' = 0$ and $D_{21}D_{21}' > 0$.

The controller Σ_K is connected to the open-loop plant Σ_P by standard negative feedback, as shown in Fig. 1. The problem to be solved here is stated as follows.

Problem 1: Find a controller Σ_K with its transfer function $C(s)$ that satisfies

$$\inf\left\{ \|H_{zw}(s)\|_2^2 : C(s) \in \mathcal{P}_c \right\} \tag{2}$$

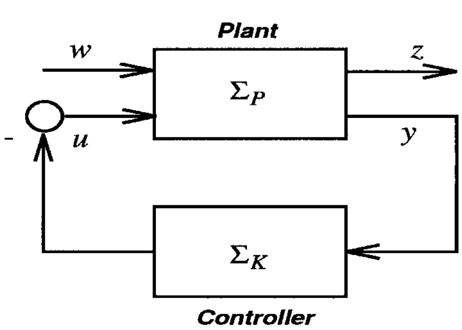

Fig. 1 Generalized plant–controller configuration.

Let us consider the following observer-based controller:

$$\Sigma_K : \begin{cases} \dot{x}_K = Ax_K + B_2 u + L(y - C_2 x_K) \\ u = -Kx_K \end{cases} \tag{3}$$

where the matrix $K \in \boldsymbol{R}^{m\times n}$ is to be determined, and $L \in \boldsymbol{R}^{n\times m}$ is given by

$$L = \Pi C_2'(D_{21}D_{21}')^{-1} \tag{4}$$

where Π is the symmetric positive-definite solution to the Riccati equation,

$$\Pi A' + A\Pi - \Pi C_2'(D_{21}D_{21}')^{-1}C_2\Pi + B_1 B_1' = 0 \tag{5}$$

Then, the $\mathcal{H}_2$ cost of $H_{zw}(s)$ is given as follows[6,7]:

$$\|H_{zw}(s)\|_2^2 = \mathrm{Tr}(C_1 \Pi C_1') + \|H_K(s)\|_2^2$$

$$H_K(s) := (C_1 - D_{12}K)[sI - (A - B_2K)]^{-1} L D_{21} \tag{6}$$

Hence, under restrictions (3–5), problem 1 is reduced to the following problem.

Problem 2: Find a controller gain $K \in \boldsymbol{R}^{m\times n}$ such that the controller Σ_K satisfies

$$\inf_K \left\{ \|H_K(s)\|_2^2 : C_K(s) \in \mathcal{P}_c \right\}$$

$$C_K(s) := K[sI - (A - B_2K - LC_2)]^{-1} L \tag{7}$$

This problem has been considered and formulated using a set of LMIs with common Lyapunov solutions in Ref. 6. To reduce the conservatism arising from seeking common Lyapunov solutions, we attempt to solve the same problem with noncommon Lyapunov solutions in this paper. Note that $\mathcal{K}_{\mathrm{SPR}}$ denotes the set of all controller gains K such that $C_K(s) \in \mathcal{P}_c$ in Eq. (7). Formulating the $\mathcal{H}_2$ optimization problem into LMIs, we derive the following problem, which is equivalent to problem 2.

Problem 3: Find a controller gain $K \in \boldsymbol{R}^{m\times n}$ such that the controller Σ_K satisfies

$$\hat{J} := \left\{ \mathrm{Tr}(C_1\Pi C_1') + \inf_{K,P_2} [\mathrm{Tr}(D_{21}'L'P_2 L D_{21})] \right\}^{\frac{1}{2}} \tag{8}$$

subject to

$$P_2 > 0$$

$$P_2(A - B_2K) + (A - B_2K)'P_2 + (C_1 - D_{12}K)'(C_1 - D_{12}K) < 0 \tag{9}$$

$$K \in \mathcal{K}_{\mathrm{SPR}} \tag{10}$$

in the matrix variable $P_2 > 0$.

When the constraint $K \in \mathcal{K}_{\mathrm{SPR}}$ is absent from problem 3, $\hat{J}$ reduces to the LQG cost, or J_{LQG}. Therefore, J_{LQG} is a theoretical lower bound on $\hat{J}$; in other words, $J_{\mathrm{LQG}} \le \hat{J}$.

III. Preliminaries

A. Positive Real and Strictly Positive Real Conditions

Let us review the following standard results.[1,8,9] Consider a linear time-invariant system Σ_G with its square transfer function $G(s) = C(sI - A)^{-1}B$.

Lemma 1. The following statements are equivalent:

1) The system Σ_G is positive real (PR).
2) $G(s)$ is analytic in $\mathrm{Re}[s] > 0$ and $G(s) + G'(s^*) \ge 0$ for $\mathrm{Re}[s] > 0$.
3) There exists $P = P' > 0$ such that $PA + A'P \le 0$ and $C = B'P$.

Lemma 2. The following statements are equivalent:

1) The system Σ_G is strictly positive real (SPR).
2) There exists $\varepsilon > 0$ such that $G(s - \varepsilon)$ is PR.
3) There exists $P = P' > 0$ such that $PA + A'P < 0$ and $C = B'P$.

B. SPR Condition on Σ_K

Based on the third statement of Lemma 2, we can immediately state the following lemma, which provides a necessary and sufficient condition for Σ_K to be SPR.

Lemma 3 (SPR condition on Σ_K). Σ_K as in Eq. (3) is SPR if and only if the following conditions hold:

$$P_s > 0, \qquad P_s(A - B_2K - LC_2) + (A - B_2K - LC_2)'P_s < 0 \tag{11}$$

$$K = L'P_s \tag{12}$$

Note that $K \in \mathcal{K}_{\mathrm{SPR}}$ in problem 3 can be replaced by the set of Eqs. (11) and (12).

C. Schur Complement Formula

The following lemma is very useful and is often used in this paper.[10]

Lemma 4 (Schur complement formula). Equations (13–15) are equivalent:

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}' & \Phi_{22} \end{bmatrix} > 0 \tag{13}$$

$$\Phi_{22} > 0, \qquad \Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}' > 0 \tag{14}$$

$$\Phi_{11} > 0, \qquad \Phi_{22} - \Phi_{12}'\Phi_{11}^{-1}\Phi_{12} > 0 \tag{15}$$

D. Conventional Result

When Eqs. (8), (9), (11), and (12) are formulated into LMIs with the common Lyapunov solution $W := P_2^{-1} = P_s^{-1} > 0$, Ref. 6 provides the following result.

Proposition 1 (SPR/$\mathcal{H}_2$ controllers with a common Lyapunov solution): Define the symmetric matrix $Q := B_2L' + LB_2'$. If the convex optimization problem

$$J_c := \left\{\mathrm{Tr}(C_1\Pi C_1') + \inf_{W,Z}\,[\mathrm{Tr}(Z)]\right\}^{\frac{1}{2}} \tag{16}$$

subject to

$$\begin{bmatrix} W & LD_{21} \\ D_{21}'L' & Z \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} AW + WA' - Q & WC_1' - LD_{12}' \\ C_1W - D_{12}L' & -I \end{bmatrix} < 0 \tag{18}$$

$$(A - LC_2)W + W(A - LC_2)' - Q < 0 \tag{19}$$

has a solution in the matrix variables $W > 0$ and $Z > 0$, then the controller gain K is given by $K = L'W^{-1}$ such that 1) $C_K(s) \in \mathcal{P}_c$ and 2) $\|H_{zw}(s)\|_2 \le J_c$.

Although the common Lyapunov solution $P_2 = P_s$ is not inherently required by Eqs. (8), (9), (11), and (12), the common Lyapunov solution is adopted to convexify the joint problem in Proposition 1 at the expense of conservatism. In the following section, we reformulate problem 3 in a manner that allows for noncommon Lyapunov solutions $P_2 \ne P_s$.

IV. Main Result

Expanding the second inequality in Eq. (9) under assumption, we complete the square of it in terms of K and apply Lemma 4 to the positive-semidefinite quadratic terms. Problem 3 is then transformed into the following problem.

Problem 4: Find a controller gain $K \in \boldsymbol{R}^{m\times n}$ such that the controller Σ_K satisfies

$$\hat{J} := \left\{\mathrm{Tr}(C_1\Pi C_1') + \inf_{K,P_2}[\mathrm{Tr}(D_{21}'L'P_2LD_{21})]\right\}^{\frac{1}{2}} \tag{20}$$

subject to

$$P_2 > 0 \tag{21}$$

$$\begin{bmatrix} P_2A + A'P_2 + C_1'C_1 - P_2B_2R_2^{-1}B_2'P_2 & \left(K - R_2^{-1}B_2'P_2\right)' \\ K - R_2^{-1}B_2'P_2 & -R_2^{-1} \end{bmatrix} < 0 \tag{22}$$

$$K \in \mathcal{K}_{\mathrm{SPR}} \tag{23}$$

in the matrix variable $P_2 > 0$.

Next, we consider the constraint $K \in \mathcal{K}_{\mathrm{SPR}}$. Substituting Eq. (12) into Eq. (11), we have

$$P_s > 0, \qquad P_s(A - LC_2) + (A - LC_2)'P_s - P_sQP_s < 0 \tag{24}$$

where $Q = B_2L' + LB_2'$ as defined in Proposition 1. Because Q is a real symmetric matrix, we can always divide it into the following two symmetric matrices. One is positive semidefinite, and the other is negative semidefinite:

$$Q = Q_1 - Q_2, \qquad Q_1 \ge 0, \qquad Q_2 \ge 0 \tag{25}$$

Note that this decomposition is not unique. One good way is to minimize $\mathrm{Tr}(Q_1)$ subject to $Q_1 - Q \ge 0$ and $Q_1 \ge 0$. Substituting Eq. (25) into Eq. (24), we have

$$P_s > 0$$

$$P_s(A - LC_2) + (A - LC_2)'P_s - P_sQ_1P_s + P_sQ_2P_s < 0 \tag{26}$$

After substituting Eq. (12) into Eq. (22), we replace $K \in \mathcal{K}_{\mathrm{SPR}}$ in problem 4 by Eq. (26) to derive the following problem, which is equivalent to problem 4.

Problem 5: Find a controller gain $K \in \boldsymbol{R}^{m\times n}$ such that the controller Σ_K satisfies

$$\hat{J} := \left\{\mathrm{Tr}(C_1\Pi C_1') + \inf_{P_2,P_s}[\mathrm{Tr}(D_{21}'L'P_2LD_{21})]\right\}^{\frac{1}{2}} \tag{27}$$

subject to

$$P_2 > 0, \qquad P_s > 0 \tag{28}$$

$$\begin{bmatrix} P_2A + A'P_2 + C_1'C_1 - P_2B_2R_2^{-1}B_2'P_2 & \left(L'P_s - R_2^{-1}B_2'P_2\right)' \\ L'P_s - R_2^{-1}B_2'P_2 & -R_2^{-1} \end{bmatrix} < 0 \tag{29}$$

$$\begin{bmatrix} P_s(A - LC_2) + (A - LC_2)'P_s - P_sQ_1P_s & P_sQ_2^{\frac{1}{2}} \\ Q_2^{\frac{1}{2}}P_s & -I \end{bmatrix} < 0 \tag{30}$$

in the matrix variables $P_2 > 0$ and $P_s > 0$.

However, Eqs. (29) and (30) are not yet LMIs due to the terms $-P_2B_2R_2^{-1}B_2'P_2 \le 0$ and $-P_sQ_1P_s \le 0$, respectively. Because these terms are negative semidefinite, we cannot apply Lemma 4 to these terms to convert Eqs. (29) and (30) into LMIs. To overcome this difficulty, we consider the following lemmas. (See Ref. 11 for earlier work.)

Lemma 5. Given $N \in \boldsymbol{R}^{m\times n}$, the following inequality holds:

$$-P_2B_2R_2^{-1}B_2'P_2 \le -P_2B_2N - N'B_2'P_2 + N'R_2N \tag{31}$$

When N is set to $N = R_2^{-1}B_2'P_2$ (equality condition), this inequality reduces to an equality.

Proof: From $R_2 > 0$, we have

$$\left(N - R_2^{-1}B_2'P_2\right)'R_2\left(N - R_2^{-1}B_2'P_2\right) \ge 0 \tag{32}$$

which yields Eq. (31) directly. When N is set to $N = R_2^{-1}B_2'P_2$, this inequality reduces to an equality. This completes the proof. □

Lemma 6. Given $M \in \boldsymbol{R}^{n\times n}$, the following inequality holds:

$$-P_sQ_1P_s \le -P_sQ_1M - M'Q_1P_s + M'Q_1M \tag{33}$$

When M is set to $M = P_s$ (equality condition), this inequality reduces to an equality.

Proof: The proof is similar to that of Lemma 5. Hence, it is omitted. □

Replacing $-P_2B_2R_2^{-1}B_2'P_2 \leq 0$ and $-P_sQ_1P_s \leq 0$ by their upper bounds in Eqs. (31) and (33), respectively, we arrive at the following problem.

Problem 6: Find a controller gain $K \in \boldsymbol{R}^{m\times n}$ such that the controller Σ_K satisfies

$$J := \left\{\mathrm{Tr}(C_1\Pi C_1') + \inf_{P_2,P_s}[\mathrm{Tr}(D_{21}'L'P_2LD_{21})]\right\}^{\frac{1}{2}} \tag{34}$$

subject to

$$P_2 > 0, \qquad P_s > 0 \tag{35}$$

$$\begin{bmatrix} P_2A_N + A_N'P_2 + N'R_2N + C_1'C_1 & \left(L'P_s - R_2^{-1}B_2'P_2\right)' \\ L'P_s - R_2^{-1}B_2'P_2 & -R_2^{-1} \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} P_sA_M + A_M'P_s + M'Q_1M & P_sQ_2^{\frac{1}{2}} \\ Q_2^{\frac{1}{2}}P_s & -I \end{bmatrix} < 0$$

$$A_N := A - B_2N, \qquad A_M := A - LC_2 - Q_1M \tag{37}$$

in the matrix variables $P_2 > 0$ and $P_s > 0$.

From Lemmas 5 and 6, we have, regarding the left-hand sides (LHSs) of Eqs. (29), (30), (36), and (37),

$$\text{LHS of Eq. (29)} \leq \text{LHS of Eq. (36)} < 0$$

$$\text{LHS of Eq. (30)} \leq \text{LHS of Eq. (37)} < 0 \tag{38}$$

Therefore, the constraints (35–37) in problem 6 are sufficient to satisfy the constraints (28–30) in problem 5. In other words, the constraints of problem 6 are more restrictive than those of problem 5; thus, all feasible solutions of problem 6 are also feasible for problem 5. The reverse is not true, although if N and M in problem 6 are selected as $N = R_2^{-1}B_2'P_2^*$ and $M = P_s^*$ based on the equality conditions in Lemmas 5 and 6, a feasible solution (P_2^*, P_s^*) of problem 5 with the corresponding $\mathcal{H}_2$ cost $\hat{J}^*$ can also be feasible for problem 6. Because the constraints (35–37) are strict inequalities, they have an infinite number of feasible solutions when they are feasible, and the set of these solutions is continuous and open. Accordingly, there should be other feasible solutions (P_2, P_s) for problem 6 in a neighbor $\mathcal{P}^*$ centered at (P_2^*, P_s^*). We call this property "neighbor feasibility." In problem 6, (P_2^*, P_s^*) yields the same objective value $J = \hat{J}^*$, whereas other feasible solutions $(P_2, P_s) \in \mathcal{P}^*$ may improve it such that $J < \hat{J}^*$. However, from Eq. (38), if a feasible solution $(P_2, P_s) \in \mathcal{P}^*$ is taken far from the center (P_2^*, P_s^*), the constraints of problem 6 become even more limiting than those of problem 5, and it becomes more difficult for J to improve the cost $\hat{J}^*$.

As a result of this compromise, we obtain an optimal solution (P_2^{**}, P_s^{**}) with $J^{**} \leq \hat{J}^*$. Because $N = R_2^{-1}B_2'P_2^*$ and $M = P_s^*$ are fixed based on the preceding value (P_2^*, P_s^*), cost improvement is restricted within this range. However, by replacing (P_2^*, P_s^*) in N and M with (P_2^{**}, P_s^{**}) such that $N = R_2^{-1}B_2'P_2^{**}$ and $M = P_s^{**}$, we can seek other feasible solutions in a new neighbor $\mathcal{P}^{**}$ centered at (P_2^{**}, P_s^{**}). Based on this property, once we have a feasible solution $(P_2^{(0)}, P_s^{(0)})$ to problem 6, we can find $(P_2^{(1)}, P_s^{(1)}), (P_2^{(2)}, P_s^{(2)}), \ldots,$ one after another, while improving $J^{(i)}, i = 0, 1, 2, \ldots,$ such that $J^{(0)} \geq J^{(1)} \geq J^{(2)} \geq \cdots \geq J^{(\infty)} \geq J_{\mathrm{LQG}}$. Note that this algorithm always converges because $\hat{J}$ has a theoretical lower bound J_{LQG} as mentioned just after problem 3.

In what follows, the superscript (i) denotes the iteration number $i = 1, 2, 3, \ldots$. In problem 6, we set $J = J^{(i)}$, $P_2 = P_2^{(i)}$, $P_s = P_s^{(i)}$, and $K = K^{(i)}$ and set $N = R_2^{-1}B_2'P_2^{(i-1)}$ and $M = P_s^{(i-1)}$.

Now we are ready to state our main result, which consists of three iterative algorithms. In Subsections IV.A and IV.B, we assume that the LMI problem (16–19) in Proposition 1 is feasible for the choice of the common Lyapunov solution $W = P_2^{-1} = P_s^{-1} > 0$. In Subsection IV.C, we will deal with the case where the problem in Proposition 1 is infeasible.

A. Iterative Algorithm 1

First, taking the optimal solution of Proposition 1 as an initial condition, we propose iterative algorithm 1:

1) Solve the LMI problem (16–19) in Proposition 1 in terms of a common Lyapunov solution $W = P_2^{-1} = P_s^{-1} > 0$ to obtain $K^{(0)} = L'W^{-1}$ with an upper bound J_c. Then set $P_2^{(0)} = P_s^{(0)} = W^{-1}$, $J^{(0)} = J_c$, and $i = 1$.

2) Given $N = R_2^{-1}B_2'P_2^{(i-1)}$ and $M = P_s^{(i-1)}$, solve problem 6 in terms of $P_2^{(i)} > 0$ and $P_s^{(i)} > 0$ to obtain $K^{(i)} = L'P_s^{(i)}$ with an upper bound $J^{(i)}$.

3) If $|J^{(i-1)} - J^{(i)}| < \varepsilon$ for some $\varepsilon > 0$, for example, $\varepsilon = 1\times 10^{-5}$, then stop.

4) Set $i \to i + 1$ and return to step 2.

The following theorem characterizes the result of this algorithm.

Theorem 1: If we carry out iterative algorithm 1, problem 6 is always feasible, and a sequence of the controller gain $K^{(i)}$ is then determined by step 2 such that 1) $C_K(s) \in \mathcal{P}_c$ and 2) $\|H_{zw}(s)\|_2 \leq \cdots \leq J^{(i)} \leq J^{(i-1)} \leq \cdots \leq J^{(1)} \leq J^{(0)}$.

Proof: The inverse of the common optimal solution $W > 0$ of the LMI problem (16–19) in Proposition 1, $P_2^{(0)} = P_s^{(0)} = W^{-1} > 0$, is a feasible solution for problem 5. This solution is also feasible for problem 6 because N and M are given as $N = R_2^{-1}B_2'P_2^{(0)}$ and $M = P_s^{(0)}$ based on the equality conditions in Lemmas 5 and 6. Because we have the freedom to seek noncommon Lyapunov solutions $P_2^{(1)} \neq P_s^{(1)}$ in addition to having neighbor feasibility as mentioned earlier, problem 6 has another feasible solution $(P_2^{(1)}, P_s^{(1)}) \in \mathcal{P}^{(0)}$ with $J^{(1)} \leq J^{(0)}$. Updating N and M with $P_2^{(1)}$ and $P_s^{(1)}$, respectively, we can seek another feasible solution $(P_2^{(2)}, P_s^{(2)}) \in \mathcal{P}^{(1)}$ with $J^{(2)} \leq J^{(1)}$. This process can be repeated for $i = 1, 2, 3, \ldots,$ and the sequence $J^{(i)}$ converges because it has a theoretical lower bound J_{LQG}. Thus, problem 6 is always feasible, which ensures $C_K(s) \in \mathcal{P}_c$. This completes the proof. □

B. Iterative Algorithm 2

Next, we consider a useful extension of iterative algorithm 1. Once $K^* \in \mathcal{K}_{\mathrm{SPR}}$ has been fixed in problem 3, the set of Eqs. (8) and (9) yields the following LMI problem.

Problem 7: Given a controller gain $K^* \in \mathcal{K}_{\mathrm{SPR}}$, solve the convex optimization problem

$$\hat{J}^* := \left\{\mathrm{Tr}(C_1\Pi C_1') + \inf_{\hat{P}_2^*}\left[\mathrm{Tr}\left(D_{21}'L'\hat{P}_2^*LD_{21}\right)\right]\right\}^{\frac{1}{2}} \tag{39}$$

subject to

$$\hat{P}_2^* > 0$$

$$\hat{P}_2^*\left(A - B_2K^*\right) + \left(A - B_2K^*\right)'\hat{P}_2^* + \left(C_1 - D_{12}K^*\right)'\left(C_1 - D_{12}K^*\right) < 0 \tag{40}$$

in the matrix variable $\hat{P}_2^* > 0$.

This $\hat{J}^*$ provides the actual $\mathcal{H}_2$ cost corresponding to $K^* \in \mathcal{K}_{\mathrm{SPR}}$. If problem 7 is solved with $K^* = K^{(i-1)} \in \mathcal{K}_{\mathrm{SPR}}$ at each iteration stage in terms of $\hat{P}_2^* > 0$, and if N is reset as $N = R_2^{-1}B_2'\hat{P}_2^*$, the actual $\mathcal{H}_2$ cost is improved (rather than its upper bound), and the convergence speed of the iterative algorithm can be accelerated. From this point of view, and setting $\hat{J}^* = \hat{J}^{(i-1)}$ in problem 7, we update iterative algorithm 1 as follows, giving iterative algorithm 2:

Replace step 2 of iterative algorithm 1 by the following steps 2a and 2b.

2a) Given $K^* = K^{(i-1)} \in \mathcal{K}_{\mathrm{SPR}}$, solve problem 7 in terms of $\hat{P}_2^* > 0$.

2b) Given $N = R_2^{-1}B_2'\hat{P}_2^*$ and $M = P_s^{(i-1)}$, solve problem 6 in terms of $P_2^{(i)} > 0$ and $P_s^{(i)} > 0$ to obtain $K^{(i)} = L'P_s^{(i)}$ with an upper bound $J^{(i)}$.

The following theorem expresses the result of this enhancement.

Theorem 2: If we carry out iterative algorithm 2, problems 6 and 7 are always feasible, and a sequence of the controller gain $K^{(i)}$ is given by step 2b such that 1) $C_K(s) \in \mathcal{P}_c$ and 2) $\|H_{zw}(s)\|_2 \leq \cdots \leq \hat{J}^{(i)} \leq J^{(i)} \leq \cdots \leq \hat{J}^{(1)} \leq J^{(1)} \leq \hat{J}^{(0)} \leq J^{(0)}$.

Proof: The proof is similar to that of Theorem 1, hence, it is omitted. □

Iterative algorithms 1 and 2 iteratively improve on the $\mathcal{H}_2$-cost upper bound and the actual $\mathcal{H}_2$ cost, respectively. As far as the upper bound is concerned, it is theoretically guaranteed that iterative algorithm 1 improves on (or at least yields a result no worse than) the standard result in Proposition 1. However, the actual $\mathcal{H}_2$ cost is not guaranteed to be better because upper-bound improvement does not always translate into actual cost improvement. In contrast, it is theoretically guaranteed that iterative algorithm 2 improves on (or at least yields a result no worse than) the standard result with regard to the actual $\mathcal{H}_2$ cost.

C. Iterative Algorithm 3

Even if the LMI problem in Proposition 1 is infeasible, we may have some valid $K \in \mathcal{K}_{\mathrm{SPR}}$. In fact, ignoring Eqs. (16–18), we always have a feasible solution $W > 0$ that satisfies Eq. (19). We call this step "subalgorithm A," and the feasibility of this algorithm is explained as follows. Because $A - LC_2$ is stable, we have $Q_0 > 0$ and $W_0 > 0$ such that

$$-Q_0 := (A - LC_2)W_0 + W_0(A - LC_2)' < 0 \tag{41}$$

Multiplying Eq. (41) by a positive scalar $\rho > 0$ and defining $W := \rho W_0$, we have

$$-\rho Q_0 := (A - LC_2)W + W(A - LC_2)' < 0 \tag{42}$$

for any positive scalar $\rho > 0$. Substituting this equation into Eq. (19), we consider the following inequality:

$$\rho Q_0 > -Q \tag{43}$$

which always holds for the choice of $\rho > \rho_0 := \lambda_{\max}(-Q)/\lambda_{\min}(Q_0)$. Thus, we can always obtain some $K \in \mathcal{K}_{\mathrm{SPR}}$ for a given L in Eqs. (4) and (5). This $K \in \mathcal{K}_{\mathrm{SPR}}$ stabilizes the closed-loop system in Fig. 1 because Σ_P is PR and Σ_K is SPR. Accordingly, the transfer function $H_{zw}(s)$ is stable. From Eqs. (1) and (3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ x_K \end{bmatrix} = \begin{bmatrix} A & -B_2K \\ LC_2 & A - B_2K - LC_2 \end{bmatrix}\begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} B_1 \\ LD_{21} \end{bmatrix} w$$

$$z = [C_1 \quad -D_{12}K]\begin{bmatrix} x \\ x_K \end{bmatrix} \tag{44}$$

$H_{zw}(s)$ is strictly proper; therefore, $\|H_{zw}(s)\|_2$ is well defined. We can, thus, update iterative algorithm 2 as follows, to get iterative algorithm 3:

Replace step 1 of iterative algorithm 2 by the following: Find a feasible solution $W > 0$ that satisfies Eq. (19) using subalgorithm A and set $K^{(0)} = L'W^{-1}$.

D. Discussion

If there exists a common Lyapunov solution, iterative algorithm 1, which improves an upper bound on the $\mathcal{H}_2$ cost, is the simplest approach. In this situation, iterative algorithm 2 is an enhancement of iterative algorithm 1 that directly improves the actual $\mathcal{H}_2$ cost at the expense of more calculation effort. With either of these methods, we can verify how much the $\mathcal{H}_2$ cost is improved from the preceding result for a common Lyapunov solution. These two algorithms are important mainly from a theoretical point of view. In contrast, from a practical point of view, iterative algorithm 3 is the most powerful because it is guaranteed to work even if the common Lyapunov method is infeasible.

V. Numerical Example

To demonstrate the applicability of the proposed algorithms, let us consider the example of a simply supported Euler–Bernoulli beam. (See Refs. 4 and 6 for details.)

The transverse deflection $d(p, t)$ is given by modal decomposition as

$$d(p, t) = \sum_{r=1}^{\infty} \sin(rp)q_r(t) \tag{45}$$

where the modal coordinates q_r, $r = 1, 2, \ldots,$ satisfy

$$\ddot{q}_r(t) + 2\zeta\omega_r\dot{q}_r(t) + \omega_r^2 q_r(t) = \sin(rp_a)u(t) \tag{46}$$

with $\omega_r = r^2$ and $\zeta = 0.01$. Both a point force actuator with intensity $u(t)$ and a velocity sensor are collocated at $p_a = 0.55l$, where l denotes the length of the beam and is set to $l = \pi$. Following Refs. 4 and 6, we model the first five modes and define the state as $x = [q_1 \ \dot{q}_1 \ \cdots \ q_5 \ \dot{q}_5]'$. The plant model is then given as follows:

$$A = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_5 \end{bmatrix}, \qquad A_r := \begin{bmatrix} 0 & 1 \\ -\omega_r^2 & -2\zeta\omega_r \end{bmatrix}$$

$$B_2 = C_2' = [0 \quad b_1 \quad 0 \quad b_2 \quad \ldots \quad 0 \quad b_5]', \qquad b_r := \sin(rp_a)$$

$$r = 1, 2, \ldots, 5 \tag{47}$$

The Bode plot of this plant is depicted in Fig. 2. Because the phase plot lies within ±90 deg, this plant is found to be PR. Performance

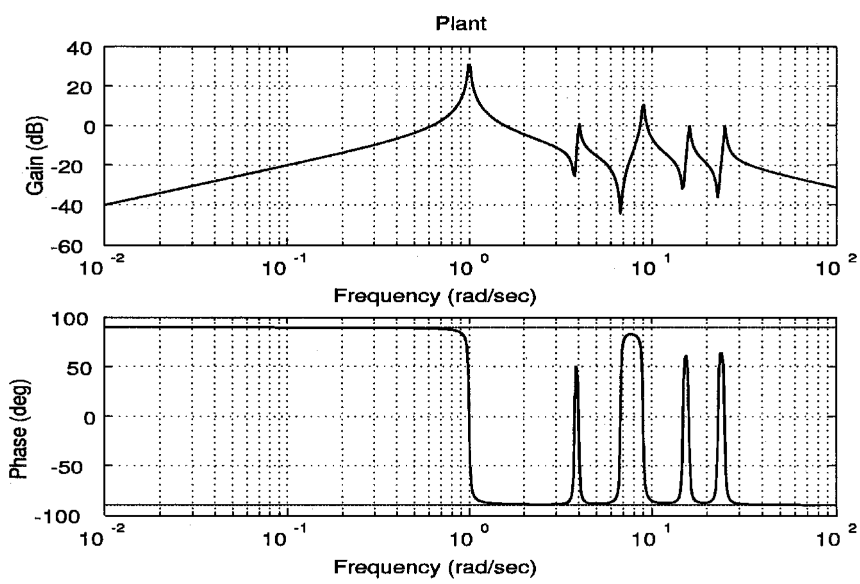

Fig. 2 Bode plot of the plant.

is defined in terms of velocity at $p_e = 0.1l, 0.2l, \ldots, 0.9l$, which yields

$$C_1 = \begin{bmatrix} 0 & c_1 & 0 & c_2 & \ldots & 0 & c_5 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \qquad c_r := \sin(rp_e)$$

$$D_{12} = \begin{bmatrix} 0 \\ 1.9 \end{bmatrix}, \qquad r = 1, 2, \ldots, 5 \tag{48}$$

We now provide two design examples. The first one is designed by iterative algorithms 1 and 2, whereas the second one is designed by iterative algorithm 3. The disturbance weight matrices B_1 and D_{21} are given as follows: For example 1,

$$B_1 = [B_2 \quad 0], \qquad D_{21} = [0 \quad 1.9] \tag{49a}$$

and for example 2,

$$B_1 = 2[B_2 \quad 0], \qquad D_{21} = [0 \quad 1.9] \tag{49b}$$

A. Example 1

From Eqs. (4) and (5), the observer gain L is calculated as follows:

$$L = [-0.0027 \quad 0.5000 \quad -0.0030 \quad -0.0481 \quad -0.0008 \quad -0.3090 \quad 0.0009 \quad 0.0827 \quad 0.0007 \quad 0.0925]' \tag{50}$$

First, we consider the case of $p_e = 0.7l$, which is the same example as in Refs. 4 and 6. In this case, the pure $\mathcal{H}_2$ controller without the SPR constraint, namely, the LQG controller, is not SPR (see Fig. 3, dashed line) and is given by

$$K_{\text{LQG}} = [0.0007 \quad 0.4060 \quad 0.1160 \quad -0.3050 \quad -0.1227 \quad -0.0563 \quad -0.1555 \quad 0.0816 \quad 0.4894 \quad 0.1736] \tag{51}$$

The LQG cost is calculated as $\|H_{zw}(s)\|_2 = 1.9843$, which gives a theoretical lower bound (for any $\mathcal{H}_2$ cost) of all of the SPR/$\mathcal{H}_2$ controllers determined for $p_e = 0.7l$ in example 1.

When we seek the common Lyapunov solution based on Proposition 1, we have an SPR controller with

$$K^{(0)} = [-0.0102 \quad 1.2764 \quad -0.4575 \quad -0.2182 \quad -0.2406 \quad -0.0931 \quad 0.1748 \quad 0.0702 \quad 1.2302 \quad 0.0868] \tag{52}$$

which yields an upper bound on $\|H_{zw}(s)\|_2 \leq 2.2258$. With this $K^{(0)}$, the actual $\mathcal{H}_2$ cost is found to be $\|H_{zw}(s)\|_2 = 2.2163$.

With iterative algorithm 1, the controller gain $K^{(i)}$ and the $\mathcal{H}_2$-cost upper bound $J^{(i)}$ are updated iteratively, as shown in Fig. 4, in which the corresponding actual $\mathcal{H}_2$ costs are also given. For $i = 4$, the algorithm stops, given the convergence tolerance $\varepsilon = 1 \times 10^{-5}$, and we have another SPR controller (see Fig. 3, solid line) with

$$K^{(4)} = [0.0004 \quad 0.4067 \quad -0.0615 \quad -0.2976 \quad -0.1270 \quad -0.0563 \quad -0.0508 \quad 0.0811 \quad 0.5478 \quad 0.1735] \tag{53}$$

This controller yields an upper bound on $\|H_{zw}(s)\|_2 \leq 1.9844$. Note that the sign has been changed in the first and seventh elements of K. With this $K^{(4)}$, the actual $\mathcal{H}_2$ cost is also found to be $\|H_{zw}(s)\|_2 = 1.9844$. This is a significant improvement that has almost matched the LQG cost.

To examine the applicability of the proposed method further, we perform similar calculations for a variety of performance points of $p_e = 0.1l, 0.2l, \ldots, 0.9l$. These results are included in Table 1, which shows how each upper bound and the actual $\mathcal{H}_2$ cost have been improved after four iterations.

Next, we show time responses of the closed-loop systems determined by $K^{(0)}$ and $K^{(4)}$. Figures 5 and 6 show the time responses of the performance output $z_1(t)$ and $z_2(t)$ ($K^{(0)}$, dashed line, and $K^{(4)}$, solid line) for $p_e = 0.1l$ and $0.7l$, respectively. Note that $z_1(t)$ is related to the state vector $\mathbf{x}(t)$ through the weight C_1, whereas $z_2(t)$ is related to the control input $u(t)$ through the weight D_{12}. The factor of 1.9 in D_{12} is selected to be consistent with the previous results

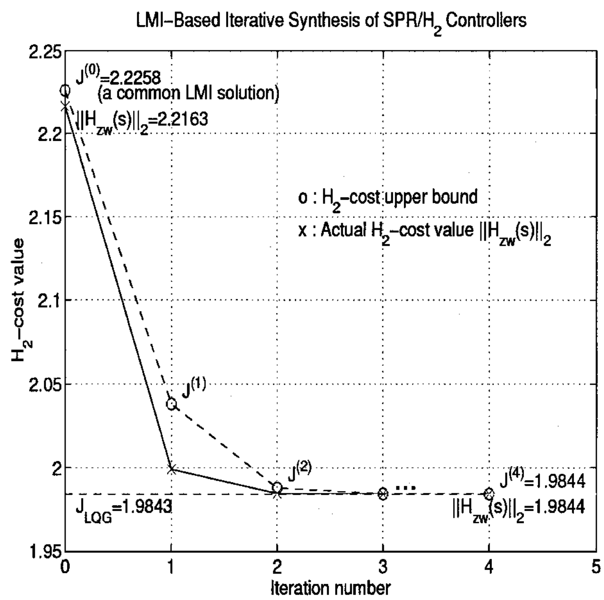

Fig. 4 $\mathcal{H}_2$-cost values calculated by iterative algorithm 1 for $p_e = 0.7l$.

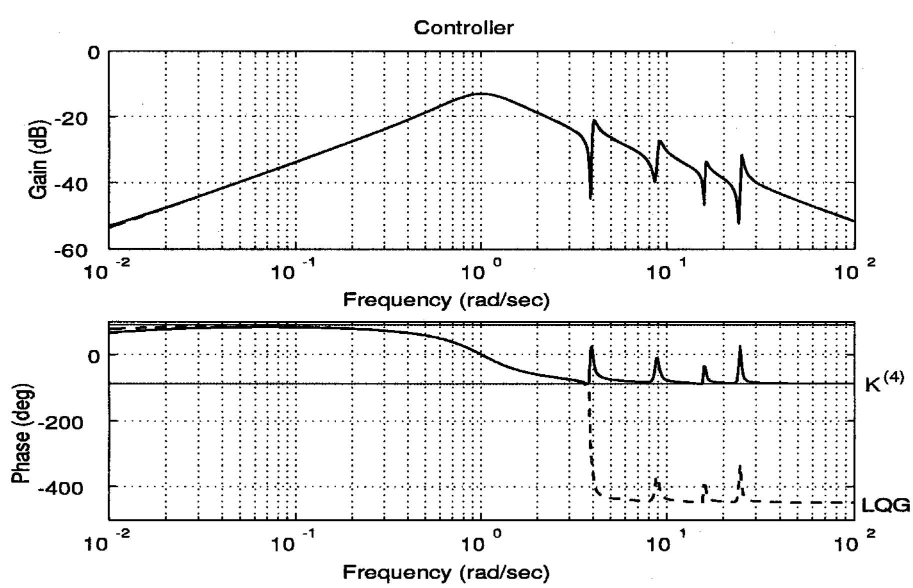

Fig. 3 Bode plot of the LQG and SPR/$\mathcal{H}_2$ controllers for $p_e = 0.7l$.

in Refs. 4 and 6 for comparison. Both time responses are simulated with an initial impulse disturbance $w(0) = [1\ 0]'$. Because the $\mathcal{H}_2$ cost of a common Lyapunov function was close enough to the LQG cost in this numerical example, we see no significant difference between the two lines for $z_1(t)$ in Figs. 5 and 6. However, we find from $z_2(t)$ that less control effort is required for the closed-loop system designed by the proposed method. (This is particularly true during the first 5 s shown in Figs. 5 and 6.) To give the reader a clearer idea of this result, the frequency responses of the closed-loop system for $p_e = 0.7l$ are given in Figs. 7 and 8. The gain of the closed-loop system from w_1 to z_2 with $K^{(4)}$ is smaller than that with $K^{(0)}$.

Table 1 Results for a variety of performance points

p_e	Upper bound	Actual $\mathcal{H}_2$ cost
$0.1l$	1.8277	1.8271
	↓	↓
	1.7392	1.7392
		(LQG: 1.7391)
$0.2l$	2.1512	2.1494
	↓	↓
	1.9743	1.9743
		(LQG: 1.9742)
$0.3l$	2.2423	2.2345
	↓	↓
	1.9863	1.9863
		(LQG: 1.9863)
$0.4l$	2.4197	2.4101
	↓	↓
	2.1425	2.1425
		(LQG: 2.1425)
$0.5l$	2.6415	2.6399
	↓	↓
	2.3971	2.3971
		(LQG: 2.3971)
$0.6l$	2.4007	2.3955
	↓	↓
	2.1425	2.1425
		(LQG: 2.1425)
$0.7l$	2.2258	2.2163
	↓	↓
	1.9844	1.9844
		(LQG: 1.9843)
$0.8l$	2.1400	2.1388
	↓	↓
	1.9731	1.9731
		(LQG: 1.9731)
$0.9l$	1.8245	1.8240
	↓	↓
	1.7384	1.7384
		(LQG: 1.7384)

Before concluding this example, we show the result of iterative algorithm 2 for $p_e = 0.7l$ in the same problem. In this case, the controller gain $K^{(i)}$ and the $\mathcal{H}_2$-cost upper bound $J^{(i)}$ are updated iteratively with the actual $\mathcal{H}_2$ cost $\hat{J}^{(i-1)}$, as shown in Fig. 9. For $i = 3$, the algorithm stops under the convergence tolerance $\varepsilon = 1 \times 10^{-5}$, and we have another SPR controller with

$$K^{(3)} = [0.0004 \quad 0.4067 \quad -0.0660 \quad -0.2956 \quad -0.1272 \quad -0.0563 \quad -0.0546 \quad 0.0811 \quad 0.5461 \quad 0.1735] \tag{54}$$

which is almost the same as the $K^{(4)}$ in Eq. (53) determined by iterative algorithm 1, and it yields an upper bound on $\|H_{zw}(s)\|_2 \leq 1.9844$. With this $K^{(3)}$, the actual $\mathcal{H}_2$ cost is also found to be $\|H_{zw}(s)\|_2 = 1.9844$. Again, this is a significant improvement that almost matches the LQG cost (1.9843).

B. Example 2

In example 2, we set $p_e = 0.7l$. From Eqs. (4) and (5), the observer gain L is calculated as follows:

$$L = [-0.0072 \quad 1.0197 \quad -0.0083 \quad -0.1611 \quad -0.0016 \quad -0.7583 \quad 0.0027 \quad 0.2832 \quad 0.0021 \quad 0.3215]' \tag{55}$$

Again, the LQG controller is not SPR, and the LQG cost is calculated to be $\|H_{zw}(s)\|_2 = 3.6654$. Although B_1 in Eq. (49) is different from that of example 1, we have the same K_{LQG} as in Eq. (51). This shows that the separation principle of LQG controllers holds for these two examples. In example 2, the problem in Proposition 1 is infeasible for the choice of the common Lyapunov solution, hence, we could not obtain any SPR/$\mathcal{H}_2$ controller in this case using previously published methods. However, even in this case, iterative algorithm 3 yields an SPR/$\mathcal{H}_2$ controller. Using subalgorithm A, we have found a feasible solution $W > 0$ to Eq. (19). This $W > 0$ yields

$$K^{(0)} = [-0.0495 \quad 7.7257 \quad 0.0849 \quad -0.9049 \quad -0.3937 \quad -0.1910 \quad -0.1043 \quad 0.1528 \quad 2.3741 \quad 0.3171] \in \mathcal{K}_{\mathrm{SPR}} \tag{56}$$

With this $K^{(0)}$, the actual $\mathcal{H}_2$ cost is found to be $\|H_{zw}(s)\|_2 = 6.0494$. By the use of iterative algorithm 3, the controller gain $K^{(i)}$ and the $\mathcal{H}_2$-cost upper bound $J^{(i)}$ are updated iteratively with the actual $\mathcal{H}_2$ cost $\hat{J}^{(i-1)}$, as shown in Fig. 10. For $i = 4$, the algorithm stops under the convergence tolerance $\varepsilon = 1 \times 10^{-5}$, and we have another SPR controller with

$$K^{(4)} = [0.0002 \quad 0.4063 \quad -0.0330 \quad -0.2999 \quad -0.1285 \quad -0.0563 \quad -0.0514 \quad 0.0814 \quad 0.5427 \quad 0.1736] \tag{57}$$

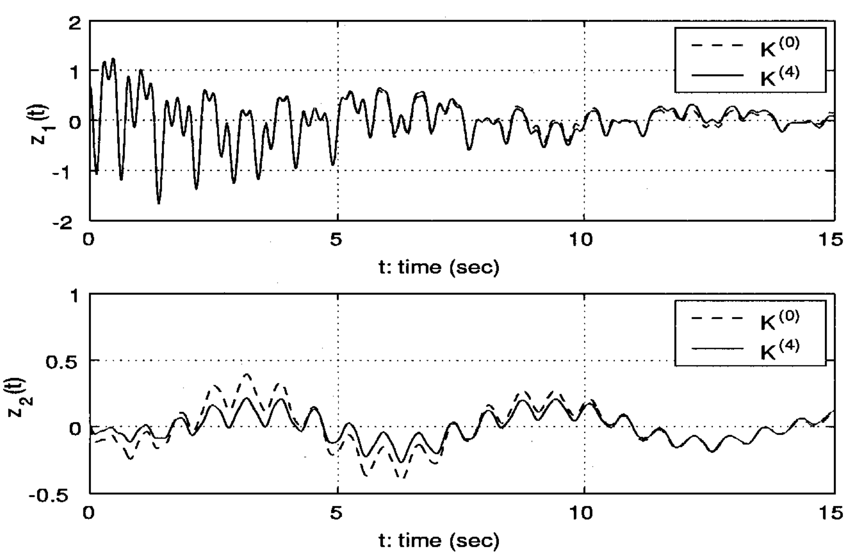

Fig. 5 Time response of the performance output for $p_e = 0.1l$.

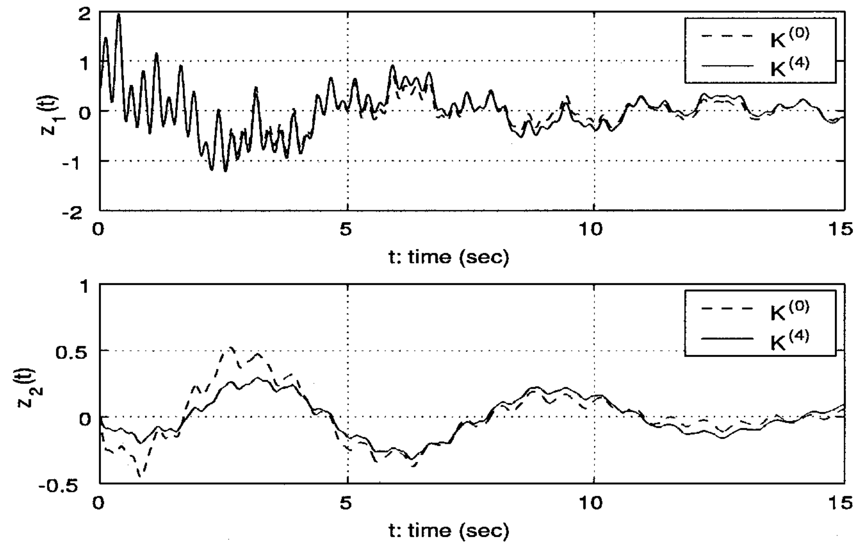

Fig. 6 Time response of the performance output for $p_e = 0.7l$.

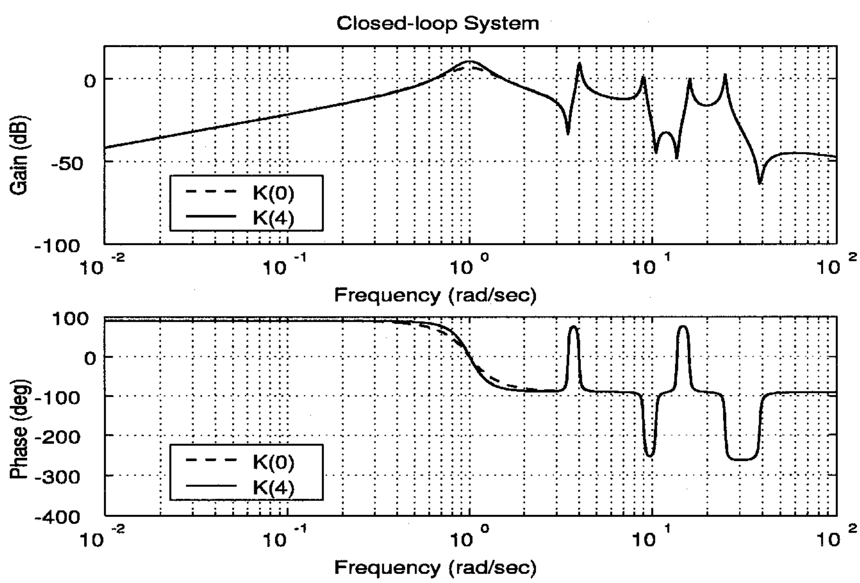

Fig. 7 Frequency response of the performance output from w_1 to z_1 for $p_e = 0.7l$.

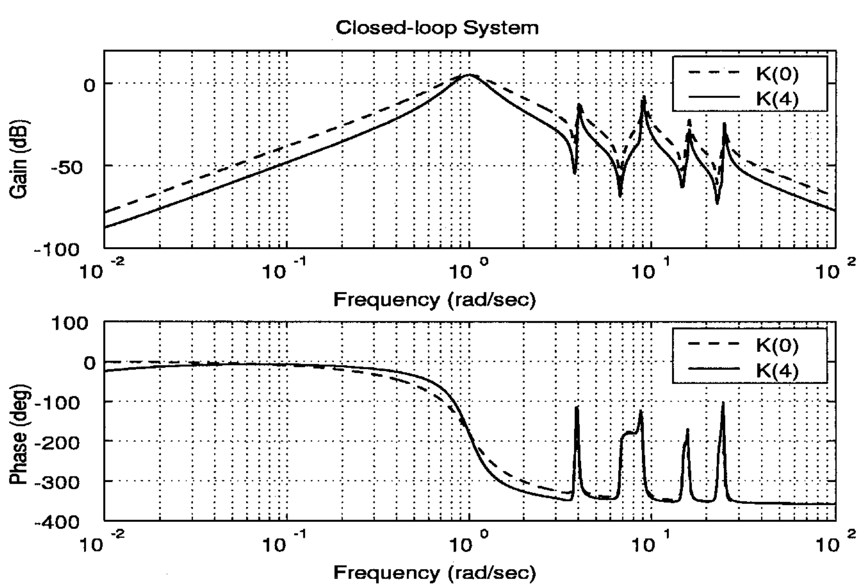

Fig. 8 Frequency response of the performance output from w_1 to z_2 for $p_e = 0.7l$.

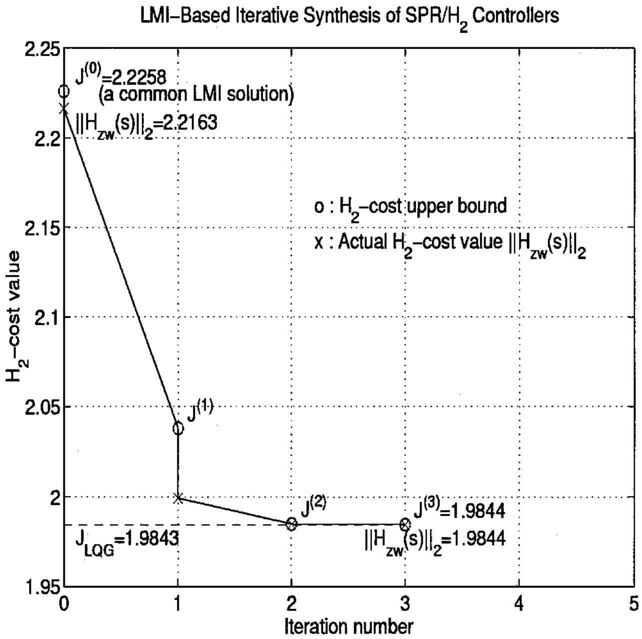

Fig. 9 **$\mathcal{H}_2$-cost values calculated by iterative algorithm 2 for $p_e = 0.7I$.**

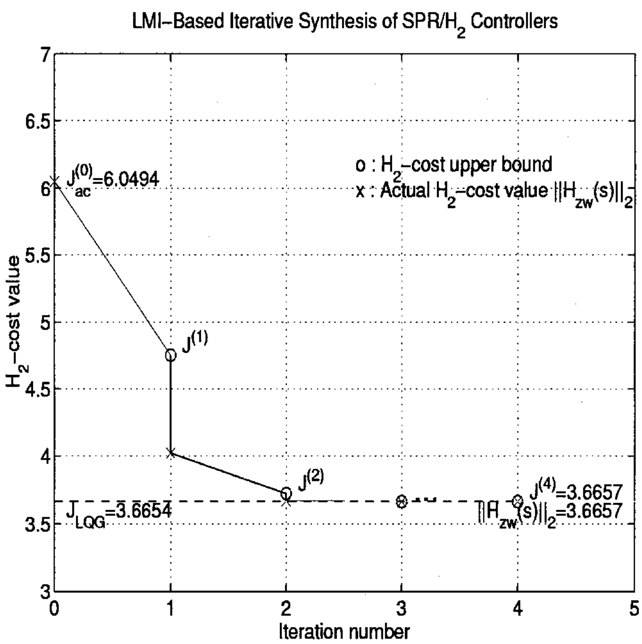

Fig. 10 **$\mathcal{H}_2$-cost values calculated by iterative algorithm 3 in example 2.**

With this $K^{(4)}$, the actual $\mathcal{H}_2$ cost is found to be $\|H_{zw}(s)\|_2 = 3.6657$, which, again, is a significant improvement that almost matches the LQG cost.

This last example highlights the potential of the proposed method. Even if the LMI problem (16–19) in Proposition 1 is infeasible from the point of view of the common Lyapunov solution, we can obtain an SPR/$\mathcal{H}_2$ controller that may effectively reduce the closed-loop $\mathcal{H}_2$ cost.

VI. Conclusions

To enhance the collocated control of large space structures, we consider the synthesis of SPR/$\mathcal{H}_2$ controllers in this paper. To reduce the conservatism of the existing technique of deriving common Lyapunov solutions, we have proposed a new synthesis technique that allows for noncommon Lyapunov solutions. In this context, three iterative algorithms have been provided. If the conventional technique of common Lyapunov solutions is feasible, iterative algorithms 1 and 2 can be used. They iteratively improve the $\mathcal{H}_2$-cost upper bound and the actual $\mathcal{H}_2$ cost, respectively. In contrast, iterative algorithm 3 can be applied even if the conventional technique of common Lyapunov solutions is infeasible. This feature is very attractive because, to date, there has been no way to obtain any SPR/$\mathcal{H}_2$ controller in the LMI-based convex optimization framework in this case. With regard to the actual $\mathcal{H}_2$ cost, iterative algorithms 2 and 3 theoretically improve on (or at least yield a result no worse than) the initial value under the constraint of controller strict positive realness. Through two illustrated examples, these algorithms have been demonstrated to improve the $\mathcal{H}_2$ performance beyond that given by the conventional technique of common Lyapunov solutions.

Acknowledgments

This work was supported by the Ministry of Education of Japan under Grant-in-Aid 12750405. The key component of this work was performed when the first author was in residence at Stanford University from October 1998 to September 1999. The first author thanks Bradford W. Parkinson, Stephen Boyd, and their colleagues for their advice and their support during his stay. He also thanks Takao Fujii at Osaka University for his constant support and encouragement.

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